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1990 J. Phys. A: Math. Gen. 23 L1079

(http://iopscience.iop.org/0305-4470/23/21/002)

J. Phys. A: Math. Gen. 23 (1990) L1079-L1083. Printed in the UK

LETTER TO THE EDITOR

The operator content of the exactly integrable SU(N) magnets

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Received 17 July 1990

Abstract. The operator content of the exactly integrable models associated with the SU(N) Lie algebras, with periodic boundaries, are studied. Our results were obtained from a numerical and analytical analysis of the associated Bethe-ansatz equations. The resulting spectra indicate that the ferromagnetic and antiferromagnetic models have different operator content for $N>2$, being realizations of distinct modular invariant solutions.

Conformal invariance has recently emerged as one of the most relevant concepts in the study of the critical properties of $(1+1)$-systems [1-3]. The possible universality classes of critical behaviour are labelled by a dimensionless number c, which is the central charge of the associated (Virasoro) algebra [1-3]. An important method for testing the predictions for conformal invariance are the exactly integrable models [4]. It is known that associated for each p $(1, 2, 3, \ldots)$ representation of the ADE Lie algebras there exists an anisotropic exactly integrable quantum chain [4]. The spectrum of these Hamiltonians, with L sites, associated with a given algebra of rank r can be block diagonalized into disjoint sectors labelled by the numbers N_a of particles of colour a $(1, 2, 3, \ldots, r)$. The eigenenergies in a given sector characterized by the set $\{N_a\}$ are given by:

$$E = \varepsilon \sum_{j=1}^{m_1} \frac{\sin(p\gamma)}{\cos(p\gamma) - \cosh(2\lambda_j^1)} \tag{1}$$

where $\varepsilon = +1$ (-1) for the ferromagnetic (antiferromagnetic) models and λ_j^a ($a = 1, 2, \ldots, r$; $j = 1, 2, \ldots, N_a$) are the solutions of the associated Bethe-ansatz equations

$$(f_{(\Omega,\alpha_a)}(\lambda_i^a))^L = -\prod_{b=1}^{r} \prod_{j=1}^{m_b} f_{(\alpha_a,\alpha_b)}(\lambda_i^a - \lambda_j^b) \tag{2}$$

where

$$f_a(x) = \frac{\sinh \gamma(x - iap/2)}{\sinh \gamma(x + iap/2)}. \tag{3}$$

The simple roots of the underlying ADE Lie algebra are denoted by α_a $(a = 1, 2, \ldots, r)$ and the highest weight of the representation by Ω.

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0305-4470/90/211079+05$03.50

The calculation of the central charge of these integrable models [5] leads to the conjecture that they are the lattice realizations of quantum field theories satisfying a Kac-Moody algebra [6] with central charge $c = p \dim G/(h+p)$ where h is the dual Coxeter number of the associated group G. The calculation of the full operator content for these models is in general more difficult. Some partial results are known for the models related with the fundamental representation ($p=1$) of the ADE Lie algebras [7] and in the SU(2) case the full operator content was calculated for arbitrary representations [8-10]. In this letter we present, for periodic chains, a general study of the full operator content of the integrable models associated with the fundamental representations of the ADE Lie algebras. We verify that in the antiferromagnetic regime ($\varepsilon = -1$) there exist, in opposition to early results [7], selection rules where many dimensions associated with excited states are forbidden in the operator content of these models. We also verify that the topology of zeros of the associated Bethe-ansatz equations are in general different in the ferromagnetic ($\varepsilon = +1$) and antiferromagnetic ($\varepsilon = -1$). These results are unexpected since these differences and selection rules do not occur in the SU(2) case [8, 9].

We concentrate most of our analysis on the exactly integrable SU(N) Heisenberg chains, which are associated with the A series of Lie algebras. The Hamiltonian of these SU(N) models is given by [4]

$$H_N = \varepsilon \sum_{l=1}^{L} \left(\sum_{\substack{i,j=1 \\ i\neq j}}^{N} Z_l^{ij} Z_{l+1}^{ji} + \cos(\gamma) \sum_{i=1}^{N} Z_l^{ii} Z_{l+1}^{ii} + \mathrm{i} \sin(\gamma) \sum_{i,j=1}^{N} \mu_{ij} Z_l^{ii} Z_{l+1}^{jj} \right) \tag{4}$$

where $Z_{k,l}^{ij} = \delta_{ik}\delta_{jl}$, $\mu_{ij} = \mathrm{sign}(i-j) + (i-j)/N$; $i, j = 1, 2, \ldots, N$ and $0 \leqslant \gamma \leqslant \pi$. The eigenenergies of (4) are given in terms of the zeros $\{\lambda_j^a\}$ of (2), where now Ω and α_a ($a = 1, 2, \ldots, N-1$) are the highest weight and roots of the SU(N) algebra. Our analytical and numerical analyses of (2) show us that the picture of zeros $\{\lambda_j^a\}$ associated with the ground state is different for the antiferromagnetic ($\varepsilon = -1$) and ferromagnetic ($\varepsilon = 1$) regimes. In the $\varepsilon = -1$ case these zeros $\{\lambda_j^a\}$ are real, forming a sea of particles†, while in the $\varepsilon = +1$ case the zeros $\{\lambda_j^a\}$ with $a = 2, 4, \ldots$ are real (particles) and those with $a = 1, 3, \ldots,$ have an imaginary part $\pi/2\gamma$ (antiparticle†).

The conformal anomaly as well as the anomalous dimensions of the operators governing the critical fluctuations can be calculated from the large-L behaviour of the eigenenergies [3]. We calculated these corrections, for the ferromagnetic regime, by using an efficient analytical method [11]. These corrections in the antiferromagnetic case were calculated previously [7]. The central charge for the both cases ($\varepsilon = \pm 1$) is $c = N-1$ and the conformal dimensions in the sector labelled by $\boldsymbol{n} = (n_1, n_2, \ldots, n_{N-1})$ have the general structure

$$X(\boldsymbol{n}, \boldsymbol{m}) = \frac{x_p}{2} \sum_{i,j=1}^{N-1} n_i C_{ij} n_j + \frac{1}{8x_p} \sum_{i,j=1}^{N-1} m_i (C^{-1})_{ij} m_j \tag{5}$$

where C is the SU(N) Cartan matrix and in the antiferromagnetic (ferromagnetic) case $x_p = (\pi - \gamma)/2\pi$ ($x_p = \gamma/2\pi$). The vectors $\boldsymbol{n} \equiv (n_1, n_2, \ldots, n_{N-1})$ and $\boldsymbol{m} \equiv (m_1, m_2, \ldots, m_{N-1})$, with $n_i, m_i \in Z$, characterize the dimensions and are generalizations of the spin-wave and vorticity numbers occurring in the SU(2) case [8].

† We denote by particle a real root λ_j^a of the Bethe-ansatz equations and by antiparticle a root with imaginary part $\pi/2\gamma$.

However by using only the analytical method above we cannot, in a closed form, make any conclusions about the operator content because the method is based on an *a priori* assumption of the existence of an excited state characterized by the vectors $(\boldsymbol{n}, \boldsymbol{m})$. In order to verify which excitations do exist we diagonalized numerically the Hamiltonian (4), for small chains, and compared it with the numerical solutions of the Bethe-ansatz equations (2). In table 1 we show some of our numerical results for the lowest dimensions in the ferromagnetic and antiferromagnetic SU(3) chains. Our numerical analysis show us that in the antiferromagnetic case the ground state is associated in (5) with the vectors $(\boldsymbol{n}, \boldsymbol{m}) = (0, 0; 0, 0)$ while the lowest non-zero dimension is associated with $(\boldsymbol{n}, \boldsymbol{m}) = (1, 1; 1, -1)$ which means that, for example, the excitation corresponding to $(\boldsymbol{n}, \boldsymbol{m}) = (1, 1; 0, 0)$ is forbidden in the antiferromagnetic case. On the other hand, in the ferromagnetic case, the lowest excitation also occurs in the sector labelled by $n_1 = n_2 = 1$ but is now associated with $(\boldsymbol{n}, \boldsymbol{m}) = (1, 1; 0, 0)$, and consequently the cases $\varepsilon = \mp 1$ have different operator contents. This difference is a consequence of the distinct topologies of the zeros of the Bethe-ansatz equations. In the ferromagnetic (antiferromagnetic) chain the excitations occur above a background of particles (particles/antiparticles).

Table 1. Finite-size estimators associated with several scaling dimensions of the ferromagnetic (X_{f}) and antiferromagnetic (X_{af}) SU(3) chain for $\gamma = \pi/4$. The exact values are given by (5).

j	$X_{\mathrm{af}}(1, 1; -1, 1)$	$X_{\mathrm{af}}(1, 1; -1, -1)$	$X_{\mathrm{af}}(1, 2; -1, 0)$
6	0.606 0876	0.936 2858	0.824 9789
24	0.597 3253	1.035 3612	1.241 6630
36	0.597 2173	1.038 9422	1.279 3576
48	0.597 1994	1.041 6622	1.297 3115
54	0.597 1978	1.041 4916	1.303 1567
extrapolated	0.597 22 (1)	1.041 66 (5)	1.347 2 (1)
exact	$0.597\,\dot{2}$	$1.041\,\dot{6}$	$1.347\,\dot{2}$

j	$X_{\mathrm{af}}(2, 2; 0, 0)$	$X_{\mathrm{f}}(1, 2; 0, 0)$	$X_{\mathrm{f}}(1, 1; 0, 0)$
6	1.375 7025	0.331 2083	0.133 8959
24	1.487 2064	0.359 8736	0.125 5639
36	1.493 7365	0.364 5911	0.125 2485
48	1.496 2510	0.367 0741	0.125 1395
54	1.496 9654	0.367 9200	0.125 1101
extrapolated	1.500 0 (2)	0.375 0 (5)	0.125 00 (4)
exact	1.5	0.375	0.125

We have also calculated numerically several excitations for SU(4) and SU(5) models. The lowest dimension is given by

$$X = \frac{1}{2}\left[\left(1 - \frac{\gamma}{\pi}\right) + \frac{N-2}{N(1-\gamma/\pi)}\right] \qquad \text{for } \varepsilon = -1 \tag{6}$$

and

$$X = \frac{1}{2}\frac{\gamma}{\pi} \qquad \text{for } \varepsilon = 1. \tag{7}$$

Again we obtain different results for the ferromagnetic and antiferromagnetic chains. It is interesting to observe that in the isotropic limit $\gamma \to 0$ the formula (6), for the $\varepsilon = -1$ regime, is precisely the lowest dimension appearing in the SU(N) Wess-Zumino-Witten-Novikov models [6]. Our numerical analysis for the excitation spectrum indicates that while in the ferromagnetic case vectors $(\boldsymbol{n}, \boldsymbol{m})$ are arbitrary with integer coordinates, in the antiferromagnetic case these vectors, for $N > 2$, have to satisfy some constraints, i.e. the integers $(n_i - m_i,\ i = 1, 2, \ldots, N-1)$ should be even numbers†. Consequently we have the result that for $N > 2$ the operator content for the antiferromagnetic and ferromagnetic chains are associated with distinct modular invariant solutions.

In the case of models associated to SU(2) algebra ($c = 1$), it has been shown [8, 9, 12] that the conformal anomaly and the operator content of other models ($c < 1$) can be obtained by continuously changing the boundary conditions. Motivated by this fact we considered the SU(N) Hamiltonian (4) with the following boundary conditions, compatible with the torus

$$
\begin{aligned}
Z^{i,j}_{L+1} &= \exp\left(\mathrm{i} \sum_{i}^{j-1} \phi_k\right) Z^{i,j}_1 \qquad & i<j \\
Z^{i,j}_{L+1} &= \exp\left(-\mathrm{i} \sum_{j}^{i-1} \phi_k\right) Z^{i,j}_1 \qquad & i>j \\
Z^{i,i}_{L+1} &= Z^{i,i}_1 \qquad & i,j = 1, 2, \ldots, N-1.
\end{aligned}
\tag{8}
$$

These boundary conditions preserve the SU(N) algebra of the matrices Z^{ij} and are specified by the angles $0 \leqslant \phi_1, \phi_2, \ldots, \phi_{N-1} \leqslant \pi$. The numerical and analytical analyses of (4) show us that analogous to the SU(2) case [8, 9, 12] the effect of these boundary conditions is to increase the vectors $\boldsymbol{m}$ in (5) and the scaling dimensions are now given by $X(\boldsymbol{n}, \boldsymbol{m} + \boldsymbol{\phi}/\pi)$, where $\boldsymbol{\phi} = (\phi_1, \phi_2, \ldots, \phi_{N-1})$. These boundary conditions correspond in the continuum model to the introduction of external charges at infinity [1], because the ground-state energy, for arbitrary $\boldsymbol{\phi}$, is related to a model with conformal anomaly $c(\boldsymbol{\phi}) = N - 1 - 12X(0, \boldsymbol{\phi}/\pi)$. If we now choose the anisotropy $\gamma = \pi/(m+1)$ $(m = 3, 4, \ldots)$ and $\phi_1 = \phi_2 = \ldots = \phi_{N-1} = 2\gamma$ we obtain

$$
c = (N-1) - \frac{N(N^2-1)}{(m+N-2)(m+N-1)} \tag{9}
$$

which is the unitary series associated with the SU(N) group [13]. Also the scaling dimensions associated with these theories [13] and generalized parafermionic theories [14] are obtained from the mass-gap amplitudes of the excited states with respect to the ground state of the $\boldsymbol{\phi}$-boundary Hamiltonian. Our analyses also show that models associated with higher representations ($p > 1$) of the SU(N) algebra, with boundary conditions (8) are related to models having conformal anomaly $c(\boldsymbol{\phi}) = [p(N^2-1)/(p+N)] - 12pX(0, \boldsymbol{\phi}/\pi)$, where now $x_p = (\pi - p\gamma)/2\pi$. Choosing $\gamma = \pi/(m+p)$ $(m = 3, 4, \ldots)$ and $\phi_1 = \phi_2 = \ldots = \phi_{N-1} = 2\gamma$ we now obtain the conformal anomaly of the general higher level SU(N) conformal series [13].

As a final remark we mention that we have also investigated the exactly integrable models associated with the O(6) and E_6 Lie algebras. In particular we note the same

† From earlier calculations [8] this restriction does not exist for the SU(2) model (XXZ chain). Also the limit $N = 2$ for the general formula (5), taking $C = 2$, does not reproduce the well known results, therefore the case $N = 2$ should be considered separately.

difference between the ferromagnetic and antiferromagnetic regimes. Our first results strongly suggest that the operator content of the models associated to the SU(N) algebra (A algebra) can be extended to the D-E series.

This work was supported in part by CNPq-Brazil, Fundacão de Amparo à Pesquisa do Estado de S Paulo (Fapesp) and by the NSF Grant PHY 86-14185. One of us (MJM) thanks V Catunda for encouragement and important comments.

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